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Some general features of matrix product states in stochastic systems*

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Abstract. We will prove certain general relations in the matrix product ansatz (MPA) for one-dimensional stochastic systems, which are true in both random and sequential updates. We will derive general MPA expressions for the currents and current correlators and find the conditions in the MPA formalism, under which the correlators are site independent or completely vanishing.

1. Introduction

One of the fruitful techniques for the study of stochastic systems on one-dimensional lattices is the matrix product ansatz (MPA) [1, 2], which is a generalization of the simple product measure, where the steady-state probabilities are represented by matrix elements or traces of a product of appropriate operators. This ansatz, when applied to processes with random sequential updates, states that the steady state of a process governed by a Hamiltonian of the form

$$H = h_1 + \sum_{i=1}^{N-1} h^B_{k,k+1} + h_N \tag{1}$$

can be written as

$$|P\rangle = \frac{1}{Z_N}\langle W|\mathcal{A}\otimes\mathcal{A}\otimes\cdots\mathcal{A}|V\rangle \tag{2}$$

where Z_N is a normalization constant. Here $\mathcal{A}$ is a column matrix with operator entries acting on some auxiliary space F, that is

$$\mathcal{A} = \sum_{i=0}^{p} A_i|i\rangle = \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ \vdots \\ A_p \end{pmatrix} \tag{3}$$

* Dedicated to Professor Ardalan, my teacher, colleague and friend; on the occasion of his 60th birthday.

and $\langle W|$ and $|V\rangle$ are two vectors in F^* and F, respectively. Here we have assumed that the Hilbert space of states of the chain is $\mathcal{H} = \boldsymbol{h}^{\otimes N}$, where $\boldsymbol{h}$ is the $(p+1)$-dimensional space of one site. The space $\boldsymbol{h}$ is spanned by the vectors $|i\rangle$; $i = 0, \ldots, p$ where $|0\rangle$ denotes a vacant site and $|i\rangle$ denotes a site occupied by a particle of type i. Note that we use the same symbol $|\,\rangle$ for a vector in $\boldsymbol{h}$ and F, hoping that no confusion will arise.

The conditions for stationarity of (2) are the following [2]:

$$h^B \mathcal{A} \otimes \mathcal{A} = \mathcal{X} \otimes \mathcal{A} - \mathcal{A} \otimes \mathcal{X} \tag{4}$$

$$(h_N \mathcal{A} - \mathcal{X})|V\rangle = 0 \tag{5}$$

$$\langle W|(h_1 \mathcal{A} + \mathcal{X}) = 0 \tag{6}$$

where $\mathcal{X}$ is a suitably chosen vector with in general operator entries, i.e.

$$\mathcal{X} = \sum_{i=0}^{p} X_i |i\rangle = \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ \vdots \\ X_p \end{pmatrix}. \tag{7}$$

When applied to processes with backward sequential (BS) updates in discrete time with an updating operator

$$\mathcal{T} = T_1 T_{12}^B T_{23}^B \cdots T_{N-1N}^B T_N \tag{8}$$

where T_1 and T_N are the boundary terms and the rest of the operators implement the bulk dynamics, the steady state $|P\rangle$ is written in the same form as in (2), but the MPA relations (4)–(6) are replaced by [3, 4]

$$T^B \mathcal{A} \otimes \hat{\mathcal{A}} = \hat{\mathcal{A}} \otimes \mathcal{A} \tag{9}$$

$$T_N \mathcal{A}|V\rangle = \hat{\mathcal{A}}|V\rangle \tag{10}$$

$$\langle W|T_1 \hat{\mathcal{A}} = \langle W|\mathcal{A} \tag{11}$$

where $\hat{A}$ is a new operator-valued vector in $\boldsymbol{h}$.

$$\hat{\mathcal{A}} = \sum_{i=0}^{p} \hat{A}_i |i\rangle = \begin{pmatrix} \hat{A}_0 \\ \hat{A}_1 \\ \hat{A}_2 \\ \vdots \\ \hat{A}_p \end{pmatrix}. \tag{12}$$

The form of the relations for forward sequential (FS) update is obtained from (9)–(11) by interchange of $\mathcal{A}$ and $\hat{\mathcal{A}}$.

This ansatz has been applied to the asymmetric simple exclusion process (ASEP) under various circumstances and with different kinds of updates (see [5–7] and references therein). To explore the usefulness of MPA beyond the above simple cases, there have also been a number of attempts to study the algebras associated with more complicated processes [8–12], or conversely, the processes associated with more complicated algebras [13–17]. In pursuing the latter line, that is beginning from algebras and searching for processes, one usually notices that the kind of algebra severely restricts the kind and the rates of the processes, especially on open systems.

The aim of this paper is to start from the general MPA algebras (4)–(6) and (9)–(11) and derive some general relations, which are valid for a large class of processes. These relations put some general constraints on the algebra that one wants to start with for the formulation of a process in the MPA formalism. We first state our main results, the proof of which will be found in the forthcoming sections.

Main results

We consider the family of processes on an open chain, described by Hamiltonians of the type (1), where there are a number of species of particles interacting in the bulk. The processes in the bulk are quite arbitrary, i.e. particles can be created, annihilated or can coagulate or decoagulate. However, we assume that some species of particles are conserved, so that for each conserved species say the ith one, a conserved current J^i can be defined. Particles are injected into or extracted from both the left and the right ends and a driving force may also be present.

Due to this arbitrariness we do not use the detailed form of any particular Hamiltonian or the associated algebra, or any of its representations thereof, but only use the basic formulae of MPA. Our main results are the following:

(a) If a species say the ith one is conserved, then its current is given by

$$\langle J^i \rangle = \frac{\langle W|X_i C^{N-1}|V\rangle}{\langle W|C^N|V\rangle}$$

where in the ordered updates $X_i = A_i - \hat{A}_i$.

(b) If $\sum_i X_i = 0$, then the n-point correlators of all the conserved currents are independent of the distances between the points. This is true even in finite open chains and is independent of what the other species are doing.

(c) If a set of $\{X_{i_1}, X_{i_2}, \ldots\}$ are c-numbers, then the connected correlation functions for the currents of particles $\{i_1, i_2, \ldots\}$ vanish in the thermodynamic limit. Moreover, in a finite chain with N sites, one has for the above set of particles

$$\langle J^r J^s \rangle_N = \langle J^s J^r \rangle_N = \langle J^r \rangle_N \langle J^s \rangle_{N-1} \tag{13}$$

$$\langle J^r J^s J^t \rangle_N = \langle J^r \rangle_N \langle J^s \rangle_{N-1} \langle J^t \rangle_{N-2} \tag{14}$$

etc.

Remarks.

1. The above properties when supplemented with the results from numerical computations or simulations for small chains, may give us an idea of what kind of algebra can or cannot be used for an exact solution of the problem in a large chain.
2. The above results are particular to open systems and do not apply to processes on closed systems.

(d) Under any circumstances the currents and densities of the conserved species in backward and forward sequential updates are related as follows:

$$\langle n^i \rangle_{\rightarrow}(k) - \langle n^i \rangle_{\leftarrow}(k) = \langle J^i{}_{\rightarrow} \rangle = \langle J^i{}_{\leftarrow} \rangle$$

where the direction of the arrows indicate the type of update. This result is a generalization of a previous one first derived in [19].

Remark. In deriving the above results we have not used any particular representation of the MPA algebra, i.e. such as the Fock space representations of [19] or [2].

2. General relations in MPA algebras

In order to deal with all the different updates in a uniform manner we define the following objects:

$$h_1 := 1 - T_1 \qquad h_N := 1 - T_N \qquad h^B := 1 - T^B \tag{15}$$

and define in the BS and FS updates the vector

$$\mathcal{X} := \mathcal{A} - \hat{\mathcal{A}}. \tag{16}$$

With these definitions, the MPA relations for these ordered updates take the following form, compared with those for the random update (see equations (1)–(4)):

$$h^B \mathcal{A} \otimes (\mathcal{A} - \mathcal{X}) = \mathcal{X} \otimes \mathcal{A} - \mathcal{A} \otimes \mathcal{X} \tag{17}$$

$$(h^N \mathcal{A} - \mathcal{X})|V\rangle = 0 \tag{18}$$

$$\langle W|(h^1 \hat{\mathcal{A}} + \mathcal{X}) = 0. \tag{19}$$

Note the similarity between the equations for different updates. In fact, equations (5) and (18) are exactly the same. This rewriting could be of no use were it not for the fact that usually (see, for example, [18, 21]), the evolution operator T^B of a process in an ordered update is, modulo a redefinition of parameters, nothing but $1 + h^B$, where h^B is the Hamiltonian of the same process in a random sequential update. Therefore, this rewriting allows one to immediately write down the MPA relations for any of these updates, once they are known for one of them. Furthermore, in those situations where a map may exist between the MPA algebras for different updates, such rewriting facilitates the search for such a map. We will see an example of this later on. Before proceeding further we fix some notation and conventions. In the Hilbert space $\boldsymbol{h}$ we define a reference state $\langle s|$ as

$$\langle s| := \sum_{i=0}^{i=p} \langle i| \equiv (1, 1, \ldots, 1). \tag{20}$$

The reference state for the space $\boldsymbol{h} \otimes \boldsymbol{h}$ is defined as $\langle ss| := \langle s| \otimes \langle s|$ and similarly for tensor products having more factors. This state is used to write the sum of entries in a column of a vector or a matrix in closed form. The following operators acting on the space F are also useful:

$$C := \sum_{i=0}^{i=p} A_i = \langle s|\mathcal{A}\rangle \qquad K := \sum_{i=0}^{i=p} X_i = \langle s|\mathcal{X}\rangle. \tag{21}$$

The normalization constant Z_N is given by $Z_N = \langle W|C^N|V\rangle$. Conservation of probabilities imply that in all three updates we should have

$$\langle ss|h^B = \langle s|h_1 = \langle s|h_N = 0. \tag{22}$$

Actually, in ordered sequential updates, conservation of probability need not hold in every single update, but only in one complete update. Therefore, one should only have $\langle S|\mathcal{T} = \langle S|$. In writing (22) we are assuming conservation of probability at each single update, i.e. $\langle ss|T^B = \langle ss|$. The known sequential updating procedures for the ASEP fall within this class.

Multiplying both sides of the MPA relations (4) and (5) or (17) and (18) (from the left by $\langle ss|$ or $\langle s|$ where appropriate and using (21) and (22) we obtain the following relations which are valid in all kinds of updates

$$K|V\rangle = \langle W|K = 0 \qquad [K, C] = 0. \tag{23}$$

This relation was first obtained in [19].

We now consider a processes in which some of the particle species are conserved locally. In this case the local operator $h^B_{k,k+1}$ does not change the number of particles in a conserved species say the ith one, on the pair of sites k and $k+1$. Thus we have

$$[\hat{\tau}^i_k + \hat{\tau}^i_{k+1}, h^B_{k,k+1}] = 0 \tag{24}$$

where $\hat{\tau}^i$ is the number operator of particles of type i, i.e. $\hat{\tau}^i|j\rangle = \delta_{ij}|j\rangle$. Applying the state $\langle ss|\tau^i_k + \tau^i_{k+1}$ on both sides of (4) or (17) and using (21) and (22) we obtain

$$[X_i, C] = [A_i, K] \qquad \forall i. \tag{25}$$

This relation holds for each conserved species separately and is a new constraint on the MPA algebra for such processes.

Remark. In a process consisting only of exchange of particles, more detailed relations can be obtained. However, equation (25) is valid independent of the interaction of the other species.

3. The MPA expression for the currents

The average density of particles of type i at site k is defined as $\langle\tau^i_k\rangle(t) := \langle S|\hat{\tau}^i_k|P(t)\rangle$ where $\langle S|$ is the reference state of the whole lattice. We will now obtain general MPA expressions for currents of conserved species of particles.

At least in the RS update, the MPA expressions for the current of a conserved species can be derived simply by calculating the current at one of the boundary links, say the rightmost link. This current is the product of the density at the rightmost site and the extraction rate of that species. However, we prefer to follow a different approach and calculate the currents directly in the bulk, in order to also calculate the current correlators (see section 3.3).

3.1. The RS update

According to the Hamiltonian formulation of Markov processes, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\tau^i_k\rangle = \langle S|[H, \hat{\tau}^i_k]|P(t)\rangle. \tag{26}$$

Due to the form of H this can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\tau^i_k\rangle = \langle S|[h^B_{k-1,k} + h^B_{k,k+1}, \hat{\tau}^i_k]|P(t)\rangle. \tag{27}$$

Combination of (24) and (27) now yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\tau^i_k\rangle = \langle S|[h^B_{k-1,k}, \hat{\tau}^i_k]|P(t)\rangle - \langle S|[h^B_{k,k+1}, \hat{\tau}^i_{k+1}]|P(t)\rangle \tag{28}$$

which can be written as a continuity equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\tau^i_k\rangle = \langle J^i_k\rangle - \langle J^i_{k+1}\rangle \tag{29}$$

with the current of particles of type i into site k being

$$\langle J_k^i \rangle = \langle S|[h^B_{k-1,k}, \hat{\tau}_k^i]|P(t)\rangle. \tag{30}$$

To find the MPA relation for the steady-state currents we rewrite (30) as

$$\langle J_k^i \rangle = \langle W|\hat{J}_k^i|V\rangle \tag{31}$$

where $\hat{J}_k^i$ is the following operator acting on F:

$$\hat{J}_k^i = \frac{1}{Z_N}\langle S|[h^B_{k-1,k}, \hat{\tau}_k^i]|\mathcal{A}\otimes\mathcal{A}\otimes\cdots\otimes\mathcal{A}\rangle. \tag{32}$$

Note that here we are looking at $|\mathcal{A}\otimes\mathcal{A}\otimes\cdots\otimes\mathcal{A}\rangle$ as an operator-valued vector in $\mathcal{H}$. Using (21), we find

$$\hat{J}_k^i = \frac{1}{Z_N}C^{k-2}\langle ss|[h^B, 1\otimes\tau^i]|\mathcal{A}\otimes\mathcal{A}\rangle C^{N-k} =: \frac{1}{Z_N}C^{k-2}M_i C^{N-k}. \tag{33}$$

The operator M_i is calculated as follows:

$$\begin{aligned} M_i &= \langle ss|h^B(1\otimes\hat{\tau}^i) - (1\otimes\hat{\tau}^i)h^B|\mathcal{A}\otimes\mathcal{A}\rangle \\ &= -\langle ss|1\otimes\hat{\tau}^i|\mathcal{X}\otimes\mathcal{A} - \mathcal{A}\otimes\mathcal{X}\rangle \\ &= -KA_i + CX_i \end{aligned} \tag{34}$$

where in the second line we have used (4) and (22) and in the third we have used (3), (7) and (21). Thus using (23), we find

$$\langle J^i\rangle = \frac{\langle W|C^{k-2}X_iC^{N-k}|V\rangle}{\langle W|C^N|V\rangle}. \tag{35}$$

We can now use (25) and rewrite $CX_i = X_iC + KA_i - A_iK$ and move the two K's to the left and right where their action on the vectors $\langle W|$ and $|W\rangle$ vanish. In this way we can move X_i completely to the left and obtain

$$\langle J^i\rangle = \frac{\langle W|X_iC^{N-1}|V\rangle}{\langle W|C^N|V\rangle}. \tag{36}$$

This kind of relation for conserved currents has already been obtained for specific 2-species algebras [14] where all the particles are conserved. Here we derive it in a quite general form.

3.2. The BS and FS updates

In the BS update we have

$$\langle\tau_k^i\rangle(t+1) - \langle\tau_k^i\rangle(t) = \langle S|\hat{\tau}_k^i\mathcal{T}|P(t)\rangle - \langle S|\hat{\tau}_k^i|P(t)\rangle. \tag{37}$$

Using the property $\langle S|\mathcal{T} = \langle S|$, we can write (37) in the form

$$\langle\tau_k^i\rangle(t+1) - \langle\tau_k^i\rangle(t) = \langle S|[\tau_k^i, \mathcal{T}]|P(t)\rangle. \tag{38}$$

Taking the structure of $\mathcal{T}$ into account (see (8)), using the notation $\mathcal{T}^B_{k,l} := T^B_{k,k+1}\dots T^B_{l,l+1}$ (with $T^B_{N,N+1} := T_N$) and the property $\langle S|\mathcal{T}^B_{k,l} = \langle S|$, we can rewrite equation (38) as

$$\begin{aligned} \langle\tau_k^i\rangle(t+1) - \langle\tau_k^i\rangle(t) &= \langle S|[\hat{\tau}_k^i, T^B_{k-1,k}T^B_{k,k+1}]\mathcal{T}^B_{k+1,N}|P(t)\rangle \\ &= \langle S|[\hat{\tau}_k^i, T^B_{k-1,k}]\mathcal{T}^B_{k,N}|P(t)\rangle + \langle S|[\hat{\tau}_k^i, T^B_{k,k+1}]\mathcal{T}^B_{k+1,N}|P(t)\rangle \end{aligned} \tag{39}$$

where in the last term we have used $\langle ss|T^B_{k-1,k} = \langle ss|$. Using the local conservation law

$$[\hat{\tau}^i_k + \hat{\tau}^i_{k+1}, T^B_{k,k+1}] = 0. \tag{40}$$

Equation (39) can be written in the form of a continuity equation

$$\langle \tau^i_k\rangle(t+1) - \langle \tau^i_k\rangle(t) = \langle J^i_k\rangle(t) - \langle J^i_{k+1}\rangle(t) \tag{41}$$

where J^i_k is the number of i-particles which, in the interval between t and $t+1$, leave site $k-1$ and enter into site k, and is given by

$$\langle J^i_k\rangle = \langle S|[\hat{\tau}^i_k, T^B_{k-1,k}]\mathcal{T}^B_{k,N}|P(t)\rangle. \tag{42}$$

The MPA relation for the current $\langle J^i_k\rangle$ is now obtained along the same lines as in the RS update, namely

$$\langle J^i_k\rangle = \langle W|\hat{J}^i_k|V\rangle \tag{43}$$

where

$$\hat{J}^i_k = \langle S|[\hat{\tau}^i_k, T^B_{k-1,k}]\mathcal{T}^B_{k,N}|\mathcal{A}\otimes\mathcal{A}\otimes\cdots\otimes\mathcal{A}\rangle. \tag{44}$$

Using (9) and (21) we obtain

$$\begin{aligned}\hat{J}^i_k &= \langle S|[\hat{\tau}^i_k, T^B_{k-1,k}]|\mathcal{A}\otimes\mathcal{A}\otimes\cdots\underbrace{\mathcal{A}\otimes\hat{\mathcal{A}}}_{k-1,k}\cdots\mathcal{A}\rangle\\ &= C^{k-2}\langle ss|[1\otimes\hat{\tau}^i, T^B]|\mathcal{A}\otimes\hat{\mathcal{A}}\rangle C^{N-k}\\ &=: C^{k-2}M_iC^{N-k}.\end{aligned} \tag{45}$$

The operator M_i is calculated by expanding the commutator and using (9) and $\langle ss|T^B = \langle ss|$, with the result

$$\begin{aligned}M_i &= \langle ss|1\otimes\hat{\tau}^i|\hat{\mathcal{A}}\otimes\mathcal{A}\rangle - \langle ss|1\otimes\hat{\tau}^i|\mathcal{A}\otimes\hat{\mathcal{A}}\rangle\\ &= \hat{C}A_i - C\hat{A}_i.\end{aligned} \tag{46}$$

Using the fact that $\hat{C} = C - K$, $\hat{A}_i = A_i - X_i$, we rewrite M_i as $CX_i - KA_i$. Using the same argument as we did in the RS case we find

$$\langle J^i{}_{\leftarrow}\rangle = \frac{\langle W|X_iC^{N-1}|V\rangle}{\langle W|C^N|V\rangle}. \tag{47}$$

A similar manipulation shows that in the FS update the current is given by the same general form as in (47), with C replaced with $\hat{C}$. Writing $\hat{C} = C + K$, expanding $(C+K)^{(N-1)}$ and using $K|V\rangle = 0$, we find that the currents of each conserved species in these two updates are equal for arbitrary transition probabilities, i.e.

$$\langle J^i{}_{\rightarrow}\rangle = \langle J^i{}_{\leftarrow}\rangle = \frac{\langle W|X_iC^{N-1}|V\rangle}{\langle W|C^N|V\rangle}. \tag{48}$$

We have proved this relation under a more general condition as in [19], that is we do not assume $C = \hat{C}$ or equivalently $K = 0$. As we have pointed out in the introduction and will prove in the next subsection the condition $K = 0$ puts physical restrictions on the steady state, namely in this case all the current correlators become distance independent. Moreover, we find the following relation between density profiles and currents in the two updates, which is valid regardless of the bulk and boundary transition rates:

$$\langle n^i\rangle_{\rightarrow}(k) - \langle n^i\rangle_{\leftarrow}(k) = \langle J^i{}_{\rightarrow}\rangle = \langle J^i{}_{\leftarrow}\rangle \tag{49}$$

the proof of which is easy to see, once we note that

$$\begin{aligned}\langle n^i\rangle_{\rightarrow}(k) &= \frac{1}{\hat{Z}_N}\langle W|\hat{C}^{k-1}\hat{A}_i\hat{C}^{N-k}|V\rangle \\ &= \frac{1}{Z_N}\langle W|C^{k-1}(A_i - X_i)C^{N-k}|V\rangle \\ &= \langle n^i\rangle_{\leftarrow}(k) - J^i{}_{\rightarrow}.\end{aligned} \tag{50}$$

This is the generalization of a similar relation for 1-ASEP [19], which is now valid for each species separately and for arbitrary transition probabilities.

3.3. *Equal time current correlators*

In this section we consider only the RS update and find the MPA expressions for the current correlators $\langle J_k^i J_l^j\rangle$ of two kinds of particles i and j at sites k and l. For $l > k+1$ (i.e. disjoint links) we obtain a simple relation. Starting from the definition

$$\langle J_k^i J_l^j\rangle := \langle S|[h^B_{k-1,k}, \hat{\tau}_k^i][h^B_{l-1,l}, \hat{\tau}_l^i]|P(t)\rangle \tag{51}$$

and proceeding exactly along the lines which led to (35) we find

$$\langle J_k^i J_l^j\rangle_N = \frac{\langle W|C^{k-1}X_iC^{l-k-2}X_jC^{N-l+1}|V\rangle}{\langle W|C^N|V\rangle}. \tag{52}$$

The proof of this relation is detailed in the appendix. For $l = k+1$ (i.e. consecutive links) no simple relation is obtained. Several remarks are in order now.

Remarks.

(a) In the special case $K = 0$, the two-point correlator (and, in fact, all the n-point current correlators (see the appendix)), become site independent, since in this case the operators X_i commute with C and the two-point correlator can be written as

$$\langle J_k^i J_l^j\rangle_N = \frac{\langle W|X_iX_jC^{N-2}|V\rangle}{\langle W|C^N|V\rangle}. \tag{53}$$

(b) For number-valued X_i and X_j this relation implies

$$\langle J_k^i J_l^j\rangle_N = \langle J_k^j J_l^i\rangle_N = \langle J^i\rangle_N\langle J^j\rangle_{N-1} \tag{54}$$

with similar relations for higher correlators. This is, of course, a finite-size effect and in the thermodynamic limit, the currents at disjoint links are not correlated. However, this is true only when some of the X_i's are number-valued. In this way we have shown that the nature of $\mathcal{X}$, controls the current correlators in a very definite way, namely for general $\mathcal{X}$, but with $K := \sum X_i = 0$, the correlators are site independent and for number-valued $\mathcal{X}$, there is no correlation in the thermodynamic limit.

(c) No such simple relations can be obtained for the FS and BS updates even for disjoint links. This is due to the fact that in these updates, no matter how far the links are, the current operators for these links contain a common string of local operators (see (42)). Thus although the currents of the two ordered updates are equal, their correlators may not be related to each other in any simple way. In this sense the steady state of the two updates may not be physically equivalent.

4. Beyond nearest-neighbour interactions

The matrix product ansatz as formulated in [2], can be generalized to models with more general Hamiltonians†. In the following we consider a Hamiltonian with nearest- and next-nearest-neighbour interactions, although our analysis can be generalized to more non-local Hamiltonians. Consider a Hamiltonian of the form

$$H = h_{12} + \sum_{k=1}^{N-2} h^B_{k,k+2} + h_{N-1,N} \tag{55}$$

where $h^B_{k,k+2}$ acts on the three sites k, $k+1$ and $k+2$ and h_{12} and $h_{N-1,N}$ are boundary terms. Writing the steady state as in (2) one finds the following conditions for stationarity:

$$h^B \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} = \mathcal{X} \otimes \mathcal{A} - \mathcal{A} \otimes \mathcal{X} \tag{56}$$

$$(h_{N-1,N} \mathcal{A} \otimes \mathcal{A} - \mathcal{X})|V\rangle = 0 \tag{57}$$

$$\langle W|(h_{12} \mathcal{A} \otimes \mathcal{A} + \mathcal{X}) = 0 \tag{58}$$

where $\mathcal{X}$ is, in general, an operator-valued tensor in $\boldsymbol{h} \otimes \boldsymbol{h}$, i.e.

$$\mathcal{X} := \sum_{\alpha,\beta} X_{\alpha,\beta} |\alpha, \beta\rangle \tag{59}$$

and $|\alpha\rangle$ and $|\beta\rangle$ denote the states of one site. Denoting as before $C := \langle s|\mathcal{A}\rangle$ and $K := \langle ss|\mathcal{X}\rangle$ one finds again that equation (23) is also true in this case. To find the MPA expressions for the currents we proceed as in section 4 and find:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \tau^i_k\rangle = \langle S|[h^B_{k-2,k} + h^B_{k-1,k+1} + h^B_{k,k+2}, \hat{\tau}^i_k]|P(t)\rangle. \tag{60}$$

Local conservation of particles, now implies

$$[h^B_{k-1,k+1}, \hat{\tau}^i_k] = -[h^B_{k-1,k+1}, \hat{\tau}^i_{k-1} + \hat{\tau}^i_{k+1}]. \tag{61}$$

Acting on (56) by $\langle sss|\hat{\tau}^i_k \otimes 1 \otimes 1 + 1 \otimes \hat{\tau}^i_k \otimes 1 + 1 \otimes 1 \otimes \hat{\tau}^i_k$ and using (21) and (22) we find that

$$[X^{(1)}_i + X^{(2)}_i, C] = 0 \tag{62}$$

where

$$X^{(1)}_i := \sum_j X_{ij} \qquad X^{(2)}_i := \sum_j X_{ji}. \tag{63}$$

Inserting (61) in (60), we find again a continuity equation as in (29) with the currents

$$\langle J^i_k\rangle = \langle S|[h^B_{k-2,k}, \hat{\tau}^i_k] - [h^B_{k-1,k+1}, \hat{\tau}^i_{k-1}]|P(t)\rangle. \tag{64}$$

The MPA expression for this current is obtained along the same lines as in section 4. After similar manipulations we find

$$\begin{aligned} \hat{J}^i_k = {} & \frac{1}{Z_N} C^{k-3} \langle sss|[h^B, 1 \otimes 1 \otimes \hat{\tau}^i]|\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}\rangle C^{N-k} \\ & - \frac{1}{Z_N} C^{k-2} \langle sss|[h^B, \hat{\tau}^i \otimes 1 \otimes 1]|\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}\rangle C^{N-k+1}. \end{aligned} \tag{65}$$

† The possibility of extending MPA to include more non-local interactions has also been recently noted in [20].

Expanding the commutators and using (21), (22) and (56) we find

$$\hat{J}_k^i = C^{k-2}\langle ss|(1 \otimes \tau^i + \tau^i \otimes 1)|\mathcal{X}\rangle C^{N-k} \tag{66}$$

which finally yields

$$\langle J^i \rangle = \frac{\langle W|X^{(i)}C^{N-1}|V\rangle}{\langle W|C^N|V\rangle} \tag{67}$$

where

$$X^{(i)} := \sum_\nu (X_{i,\nu} + X_{\nu,i}). \tag{68}$$

The generalization of these results to Hamiltonians with more non-local interactions is obvious.

5. Discussion

We have discussed the general structure of MPA states in stochastic systems and have proved certain general relations for general stochastic processes in random and ordered updates. We have tried to be as general as possible, our results in sections 3–5 are independent of the bulk and boundary transition rates and are also independent of the asymmetry caused by driving. We have found general MPA expressions for the currents and current correlators and have shown that when in the MPA formalism one uses general operator-valued $\mathcal{X}$ but with $\sum_i X_i = 0$, the current correlators while non-vanishing, become site independent.

And when any of the X_i's are c-numbers, then the connected correlation functions of the corresponding currents vanish in the thermodynamic limit.

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Appendix

In this appendix we present the proof of (65). In the same spirit that we have derived the expressions for the currents we write

$$\langle J_k^i J_l^j \rangle = \langle W|\hat{J}_k^i \hat{J}_l^j|V\rangle \tag{A1}$$

where

$$\begin{aligned}\hat{J}_k^i \hat{J}_l^j &= \frac{1}{Z_N}\langle S|[h_{k-1,k}^B, \hat{\tau}_k^i][h_{l-1,l}^B, \hat{\tau}_l^i]|\mathcal{A} \otimes \mathcal{A} \cdots \otimes \mathcal{A}\rangle \\ &= \frac{1}{Z_N} C^{k-2} M_i C^{l-k-2} M_j C^{N-l} \\ &= \frac{1}{Z_N} C^{k-2}(CX_i - KA_i)C^{l-k-2}(X_j C - A_j K)C^{N-l}\end{aligned} \tag{A2}$$

where in the last line we have used two equivalent expressions for M, in order to move the two K's to the left and right, respectively, and act by them on $\langle W|$ and $|V\rangle$ to obtain

$$\langle J_k^i J_l^j \rangle = \frac{\langle W|C^{k-1}X_i C^{l-k-2} X_j C^{N-l+1}|V\rangle}{\langle W|C^N|V\rangle}. \tag{A3}$$

This result is true for the two-point correlators, since in higher correlators one cannot eliminate all the K's in (A2). When $K = 0$, this further simplifies to (53). Furthermore, in this case formula (53) trivially generalizes to n-point functions.

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